Simulation of Magneto-Hydrodynamics with Least-Squares Finite Element Method

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Overview

- Introduction
- Governing Equations
- General LSFEM formulation
- LSFEM for MHD
- Applications of LSFEM for MHD
- LSFEM for EMHD
- Application of LSFEM for EMHD

Introduction

- EMHD is study of incompressible flow under the influence of electric and magnetic fields
- Various simplified analytical models exist and have been used for numerical simulation(EHD,MHD)

Introduction cont.

- A fully consistent non-linear model for general EMHD has been recently developed
- No numerical simulations of full EMHD has been report in open literature to date
- A computer code for numerical simulation for 2-D planar MHD/EMHD flows has been developed using LSFEM

Introduction cont.

- Numerical simulation is necessary for performing optimization involving EMHD flows

Applications of EMHD

- Manufacturing(solidification,crystal growth)
- Flow control
- Drag reduction/propulsion
- Pumps with no moving parts(artificial heart, liquid metal pumps)
- Compact heat exchangers
- Shock absorbers, active damping

Fully Consistent Model of EMHD

Conservation of Linear Momentum

$$
\begin{aligned}
\text{ñ}\,\frac{D\underline{v}}{Dt} = {} & -\text{ñ}g[1-\text{á}\left(\text{Ô}-\text{Ô}_0\right)]\underline{\text{i}}_3 \;\; -\nabla\left(p+p_e+p_m\right) \\
& + \nabla\cdot\left(\text{ì}_{\,v}(\nabla\underline{v}+\nabla\underline{v}^{\,t})\right) - \nabla\cdot\left(\text{ó}_2(\underline{\hat{\text{E}}}\otimes\underline{\hat{\text{E}}})\right) \\
& - \nabla\cdot\left(\frac{\hat{\text{e}}_2}{T}\left(\nabla T\otimes\nabla T\right)\right) - \nabla\cdot\left(\left(\frac{\hat{\text{e}}_5}{T}+\text{ó}_5\right)\left(\underline{\hat{\text{E}}}\otimes\nabla T\right)_S\right) \\
& + q_e\underline{\hat{\text{E}}} \;\; + \text{ó}_1\underline{\hat{\text{E}}}\times\underline{B} \;\; + \text{ó}_2\underline{\underline{d}}\cdot\underline{\hat{\text{E}}}\times\underline{B} \;\; + \text{ó}_4\nabla T\times\underline{B} \\
& + \text{ó}_5\underline{\underline{d}}\cdot\nabla T\times\underline{B} + \text{ó}_7\,(\underline{\hat{\text{E}}}\times\underline{B})\times\underline{B} + \frac{\hat{\text{e}}_{10}}{T}\,(\nabla T\times\underline{B})\times\underline{B} \\
& + \text{å}_p\left(\underline{\hat{\text{E}}}\cdot\nabla\right)\underline{E} + \frac{1}{\text{ì}_m}\left(\underline{B}\cdot\nabla\right)\underline{B} - \text{å}_p\left(\left(\underline{v}\times\underline{E}\right)\cdot\nabla\right)\underline{B} \\
& + \frac{D}{Dt}\left(\text{å}_p(\underline{\hat{\text{E}}}\times\underline{B})\right)
\end{aligned}
$$

Conservation of Energy

$$\text{ñ}C_p \frac{DT}{Dt} = Q_h + \nabla \cdot \left(\text{ê}_1 \nabla \text{Ô} + \text{ê}_2 \underline{\underline{d}} \cdot \nabla \text{Ô} + \text{ê}_4 \underline{\hat{E}}\right)$$

$$+ \nabla \cdot \left(\text{ê}_5 \underline{\underline{d}} \cdot \underline{\hat{E}} + \text{ê}_7 \nabla \text{Ô} \times \underline{B} + \text{ê}_{10} \underline{\hat{E}} \cdot \underline{B}\right)$$

$$+ \text{ó}_1 \underline{\hat{E}} \cdot \underline{\hat{E}} + \text{ó}_4 \underline{\hat{E}} \cdot \nabla \text{Ô} - \frac{\text{ê}_2}{\text{Ô}} \nabla \text{Ô} \cdot \underline{\underline{d}} \cdot \nabla \text{Ô}$$

$$- \frac{\text{ê}_5}{\text{Ô}} \underline{\hat{E}} \cdot \underline{\underline{d}} \cdot \nabla \text{Ô} + \frac{\text{ê}_{10}}{\text{Ô}} \underline{\hat{E}} \cdot \left(\nabla \text{Ô} \times \underline{B}\right)$$

$$+ \underline{\hat{E}} \cdot \frac{D\left(\text{å}_p \underline{\hat{E}}\right)}{Dt} - \left(\frac{\underline{B}}{\text{ì}_m} - \text{å}_p \underline{V} \times \underline{E}\right) \cdot \frac{D\underline{B}}{Dt} .$$

Conservation of Mass

$$\nabla \cdot \underline{v} = 0$$

Maxwell's Equations

$$\nabla \cdot \left(\text{å}\underline{E} + \text{å}_p \underline{v} \times \underline{B}\right) = q_e ,$$

$$\nabla \cdot \underline{B} = 0 ,$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} ,$$

$$\nabla \times \frac{\underline{B}}{\text{ì}} + \nabla \times (\text{å}_p \underline{v} \times \underline{E}) = \frac{\partial}{\partial t}\left(\text{å}\underline{E} + \text{å}_p \underline{v} \times \underline{B}\right) + q_e \underline{v}$$

$$+ \text{ó}_1 \underline{\hat{E}} + \text{ó}_2 \underline{\underline{d}} \cdot \underline{\hat{E}} + \text{ó}_4 \nabla \text{Ô} + \text{ó}_5 \underline{\underline{d}} \cdot \nabla \text{Ô}$$

$$+ \text{ó}_7 \underline{\hat{E}} \times \underline{B} + T^{-1} \text{ê}_{10} \nabla \text{Ô} \times \underline{B} .$$

Sub-models

The LSFEM was applied to two sub-models of the fully nonlinear Electro-magneto-hydrodynamic system

•Magneto-hydrodynamics

•Electro-magneto-hydrodynamics with reduced number of source terms

Advantages of the LSFEM

- Can use equal order basis functions for pressure and velocity
- can use the first order form of PDE’s
- can handle any type of equation and mixed types of equations
- Can discretize convection terms without upwinding or explicit artificial dissipation
- Stable and robust method
- Resulting system of equations is symmetric and positive definite

- Simple iterative techniques such as PCG and multigrid can be used to solve the system of equations
- Inclusion of divergence constraint for magnetic flux is straight forward

Least-squares finite element method (LSFEM)

The system of partial differential equations described in section 2.1 is discretized using the least squares finite element method. We first look at the LSFEM for a general linear first-order system

$$[L]\mathbf{u} = \mathbf{f} \tag{13}$$

where

$$[L] = [A_1]\frac{\partial}{\partial x} + [A_2]\frac{\partial}{\partial y} + [A_3] \tag{14}$$

The residual of the system is represented by $\mathbf{R}$.

$$\mathbf{R}(\mathbf{u}) = [L]\mathbf{u} - \mathbf{f} \tag{15}$$

We now define the following least squares functional I over the domain Ω

$$I(\mathbf{u}) = \int_\Omega \mathbf{R}(\mathbf{u})^T \cdot \mathbf{R}(\mathbf{u})\, dx\, dy \tag{16}$$

The weak statement is then obtained by taking the variation of I with respect to $\mathbf{u}$ and setting the result equal to zero.

$$\delta I(\mathbf{u}) = \int_{\Omega} \left([L]\delta\mathbf{u}\right)\left([L]\mathbf{u} - \mathbf{f}\right) dx\, dy = 0 \tag{17}$$

Using equal order shape functions, ϕ_i, for all unknowns, the vector $\mathbf{u}$ is written as

$$\mathbf{u} = \sum_{i=1}^{n} \phi_i \{u_1, u_2, u_3, ..., u_m\}_i^T \tag{18}$$

where $\{u_1, u_2, u_3, ..., u_m\}_i$ are the nodal values at the ith node of the finite element. Introducing the above approximation for $\mathbf{u}$ into the weak statement leads to a linear system of algebraic equations

$$[K]\mathbf{U} = \mathbf{F} \tag{19}$$

where $[K]$ is the stiffness matrix, $\mathbf{U}$ is the vector of unknowns, and $\mathbf{F}$ is the force vector.

LSFEM Code

A serial code was developed in C/C++ to solve general systems with LSFEM

•Steady state problems only

•Mixed triangular and quadrilateral meshes

•Quadratic interpolation functions for all unknowns

•Solve resulting systems with either sparse LU factorization or Jacobi PCG

•Support for multiple material domains such as in conjugate heat transfer problems

•Nonlinear equations are linearized with Newton or Picard method

LSFEM for MHD

The steady viscous incompressible MHD flow can be described by the Navier-Stokes equations combined with the Maxwell's equations.

$$
\begin{aligned}
\nabla \cdot \mathbf{V} &= 0 && (1)\\
\rho \mathbf{V} \cdot \nabla \mathbf{V} - \eta \nabla^2 \mathbf{V} + \nabla P - \sigma \mathbf{V} \times \mathbf{B} \times \mathbf{B} &= 0 && (2)\\
\rho C_p \mathbf{V} \cdot \nabla T - \nabla \cdot (k \nabla T) - \tfrac{1}{\sigma} \mathbf{J} \cdot \mathbf{J} &= 0 && (3)\\
\nabla \cdot \mathbf{B} &= 0 && (4)\\
\nabla \times \mathbf{B} &= \mu \mathbf{J} && (5)\\
\mathbf{J} &= \sigma \mathbf{V} \times \mathbf{B} && (6)\\
& && (7)
\end{aligned}
$$

First order system for MHD

$$
\begin{aligned}
\nabla \cdot \mathbf{V} &= 0 && (20)\\
\mathbf{V} \cdot \nabla \mathbf{V} + \tfrac{1}{Re} \nabla \times \omega + \nabla P - \tfrac{Ht^2}{Re} \mathbf{V} \times \mathbf{B} \times \mathbf{B} &= 0 && (21)\\
\omega - \nabla \times \mathbf{V} &= 0 && (22)\\
\mathbf{V} \cdot \nabla T + \nabla \cdot \mathbf{q} - \tfrac{Ht^2 Ec}{Re} (\mathbf{V} \times \mathbf{B})^2 &= 0 && (23)\\
\mathbf{q} + \tfrac{1}{Pe} \nabla T &= 0 && (24)\\
\nabla \times \mathbf{q} &= 0 && (25)\\
\nabla \cdot \mathbf{B} &= 0 && (26)\\
\nabla \times \mathbf{B} &= Rm \mathbf{V} \times \mathbf{B} && (27)
\end{aligned}
$$

First order system for 2-D MHD

$$\left[A_1^{fluid}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u_0 & 0 & 1 & 0 \\ 0 & u_0 & 0 & -\frac{1}{Re} \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad \left[A_2^{fluid}\right] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ v_0 & 0 & 0 & \frac{1}{Re} \\ 0 & v_0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\left[A_3^{fluid}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{Ht^2}{Re} B_{y0}^2 + \frac{\partial u_0}{\partial x} & -\frac{Ht^2}{Re} B_{x0} B_{y0} + \frac{\partial u_0}{\partial y} & 0 & 0 \\ -\frac{Ht^2}{Re} B_{x0} B_{y0} + \frac{\partial v_0}{\partial x} & \frac{Ht^2}{Re} B_{x0}^2 + \frac{\partial v_0}{\partial y} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{f}^{fluid} = \left\{ \begin{array}{c} 0 \\ u_0 \frac{\partial u_o}{\partial x} + v_0 \frac{\partial u_o}{\partial y} \\ u_0 \frac{\partial v_o}{\partial x} + v_0 \frac{\partial v_o}{\partial y} \\ 0 \end{array} \right\}, \qquad \mathbf{u}^{fluid} = \left\{ \begin{array}{c} u \\ v \\ p \\ \omega \end{array} \right\}$$

$$[A_1^{mag}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, [A_2^{mag}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, [A_3^{mag}] = \begin{bmatrix} 0 & 0 \\ Rm\, v_0 & -Rm\, u_0 \end{bmatrix},$$

First order system for 2-D MHD cont.

$$\mathbf{f}^{mag} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}, \qquad \mathbf{u}^{mag} = \left\{ \begin{array}{c} B_x \\ B_y \end{array} \right\}$$

$$[A_1^{heat}] = \begin{bmatrix} u_0 & 1 & 0 \\ \frac{1}{Pe} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, [A_2^{heat}] = \begin{bmatrix} v_0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{Pe} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, [A_3^{heat}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{f}^{heat} = \left\{ \begin{array}{c} \frac{Ht^2 Ec}{Re} (u_0 B_{y_0} - v_0 B_{x0})^2 \\ 0 \\ 0 \\ 0 \end{array} \right\}, \qquad \mathbf{u}^{heat} = \left\{ \begin{array}{c} T \\ q_x \\ q_y \end{array} \right\}$$

Verification of Accuracy

- No analytic solutions for EMHD exist
- Some analytic solutions for MHD exist
- NSE portion of code was validated using analytic solutions for NSE and with experiment data from driven cavity flows and backward facing step
- Heat transfer/Electric/Magnetic field portions were verified with analytic solutions

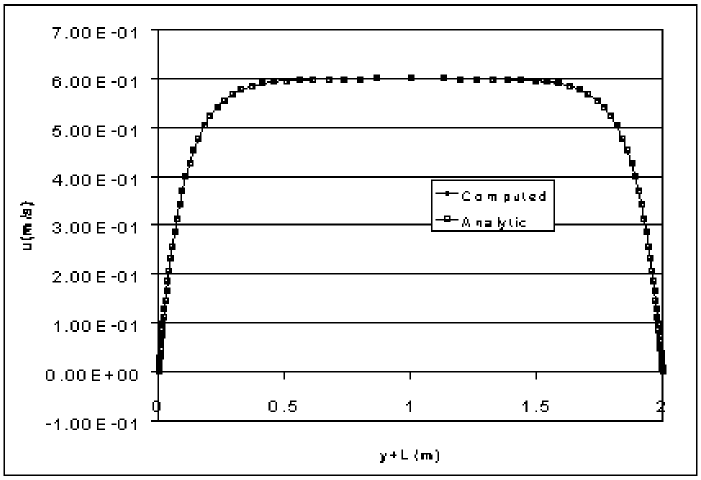
Hartmann Flow

MHD LSFEM code was compared with the analytic solution to Poisuille-Hartmann flow

Parameters for Poisuille-Hartmann flow test problem

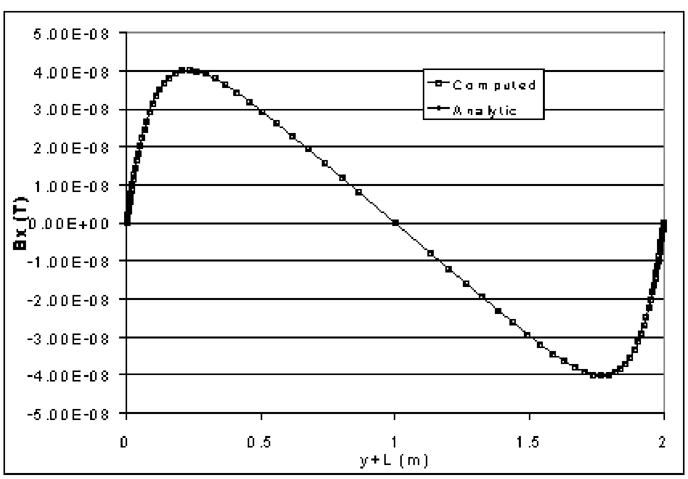
Ht	10
Rm	6×10^{-7}
$L_0\,(m)$	1
$U_0\,(m\,s^{-1})$	0.6
$\eta\,(kg\,m^{-1}s^{-1})$	0.01
$B_0\,(T)$	1
$\mu\,(H\,m^{-1})$	1×10^{-6}
$\partial P/\partial x\,(Pa\,m^{-1})$	0.6
$\sigma\,(\Omega^{-1}m^{-1})$	1

Comparison of LSFEM and analytic solution for Hartmann flow



Computed and analytic velocity profile

Comparison of LSFEM and analytic solution for Hartmann flow



Computed and analytic induced magnetic field

Simulation and Optimization of Magneto-hydrodynamic Flows with LSFEM

Optimization of Magneto-Hydrodynamic Control of Diffuser Flows Using Micro-Genetic Algorithms and Least-Squares Finite Elements

Goal

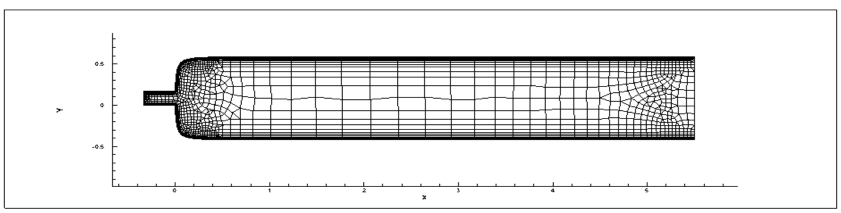
- Given a fixed diffuser shape, use micro-GA and LSFEM MHD analysis to design a magnetic field distribution on the diffuser wall that will increase static pressure rise

Flow Solver

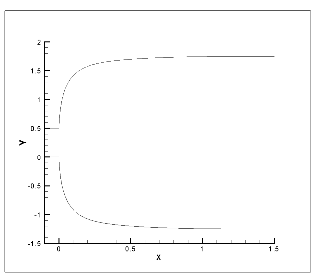
- LSFEM solver for 2-D steady incompressible Navier-Stokes together with Maxwell’s equations for steady magnetic field
- Uses hybrid quadrilateral/triangular grid
- One analysis takes around 22 min. on a single Pentium II CPU

BC’s and Parameterization

- Parabolic velocity specified at inlet
- Static pressure specified at outlet
- no-slip conditions on wall
- magnetic field component along wall are specified. They were parameterized with b-spline
- perfectly conducting wall bc used on all other solid surfaces

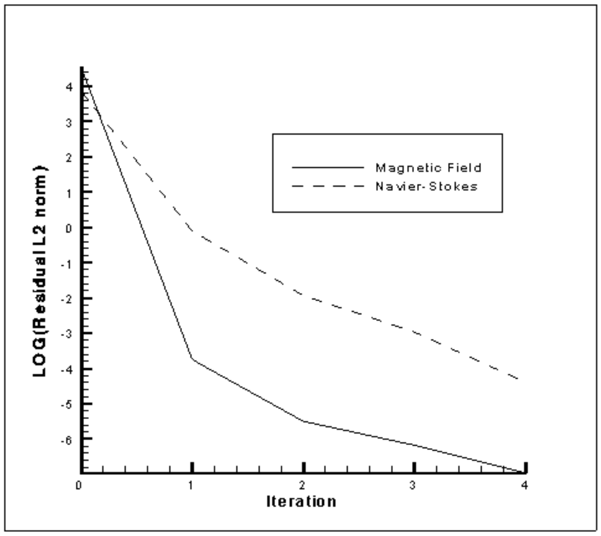


Hybrid mesh for diffuser analysis



Given diffuser shape

Convergence History for a Typical Design



Physical parameters for diffuser optimization problem

$\rho\,(kg\ m^{-2})$	1025
$L_0\,(m)$	3
$U_0\,(m\,s^{-1})$	1.58×10^{-4}
$\eta\,(kg\,m^{-1}s^{-1})$	0.001
$\mu\,(H\,m^{-1})$	1×10^{-6}
$\sigma\,(\Omega^{-1}m^{-1})$	4.5

Parallel Genetic Algorithm

- GA is a naturally coarse grained parallel algorithm
- One node maintains the population(master) and distributes jobs to the slave nodes
- Only simple synchronous message passing is needed to implement on distibuted memory
- Population size need not match the number of slave nodes
- Asynchronous models are also being developed for use when function analysis computation times vary dramatically.

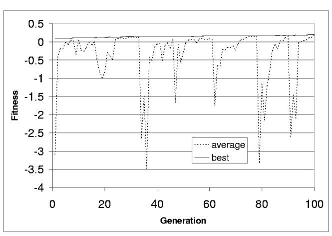
Parallel Computer

- based on commodity hardware components and public domain software
- 16 dual Pentium II 400 MHz based PC's
- 100 Megabits/second switched ethernet
- total of 32 processors and 8.2 GB of main memory
- Compressible NSE solver achieved 1.5 Gflop/sec with a LU SSOR solver on a 100x100x100 structured grid

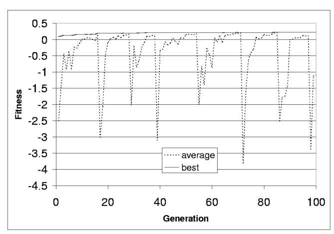
Genetic Algorithm

- Population size of 15
- 100 generations
- 9-bit strings for each design variable
- elitism
- tournament selection
- uniform crossover
- parallel micro-GA

GA Convergence History



Run 1



Run 2

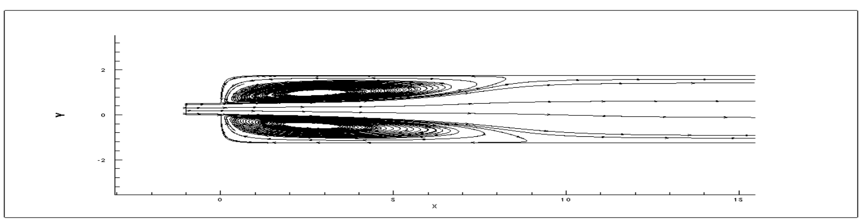
Results

Two optimizations were run simultaneously with 16 processors each.

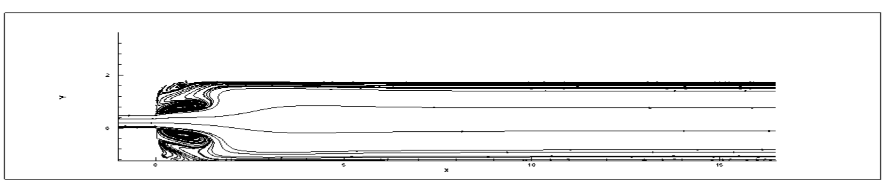
Generation 100 was reached by both in about 30 hours

run 1 achieved a pressure increase of .207 Pa
run 2 achieved a pressure increase of .228 Pa

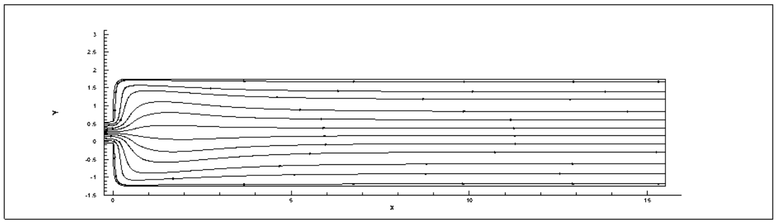
diffuser without an applied magnetic field achieved a pressure increase of .05 Pa.

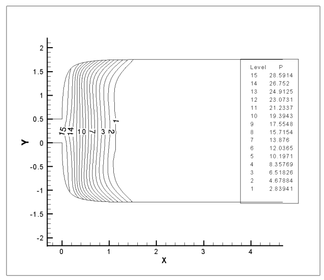


With no applied magnetic field



With optimized applied magnetic field





Simulation of Magneto-hydrodynamic Flows with Conjugate Heat Transfer with LSFEM

Objective

- Simulation of flow through channel with an applied magnetic field
- Simulation of heat transfer from the flow to a solid cold wall
- Observe the effect of applied magnetic field on flow patterns and heat transfer characteristics

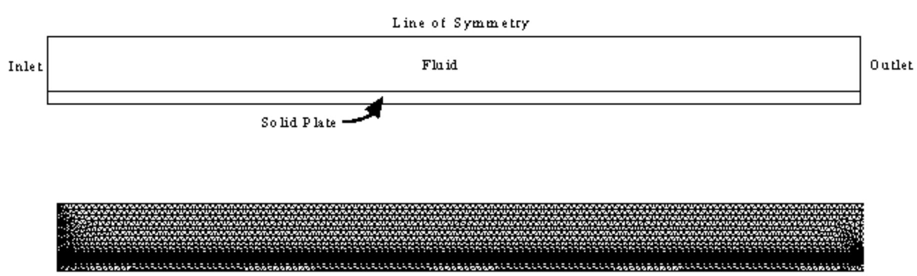


Figure 3: Computational domain and mesh for conjugate heat transfer problem

Boundary Conditions

- Inlet temperature of 2000 K
- Specified outlet pressure
- Specified parabolic velocity profile at inlet
- No-slip on walls
- Symmetry boundary condition on top
- Temperature of 300 K on bottom wall
- Perfectly conducting walls except in the region 7 < x < 8 where sinusoidal magnetic field components were specified. Magnitude was varied from 0 to 5 Tesla.

Table 2: Physical parameters for channel problem

$\rho\,(kg\ m^{-2})$	1024.0
$L_0\,(m)$	1.0
$U_0\,(m\,s^{-1})$	6.0×10^{-4}
$\eta\,(kg\,m^{-1}s^{-1})$	0.001
$\mu\,(H\,m^{-1})$	1×10^{-6}
$\sigma\,(\Omega^{-1}m^{-1})$	4.5
$Cp\,(JKg^{-1}K^{-1})$	4184.0
$k_{fluid}\,(Wm^{-1}K^{-1})$	0.5
$k_{solid}\,(Wm^{-1}K^{-1})$	10.0
$B_o\,(T)$	0.0-5.0

Results

- Presence of magnetic field induces a large separation in the flow field close to the wall
- Size and complexity are proportional to the strength of the magnetic field
- A drop in fluid/solid interface temperature was observed in the region where the magnetic field was applied

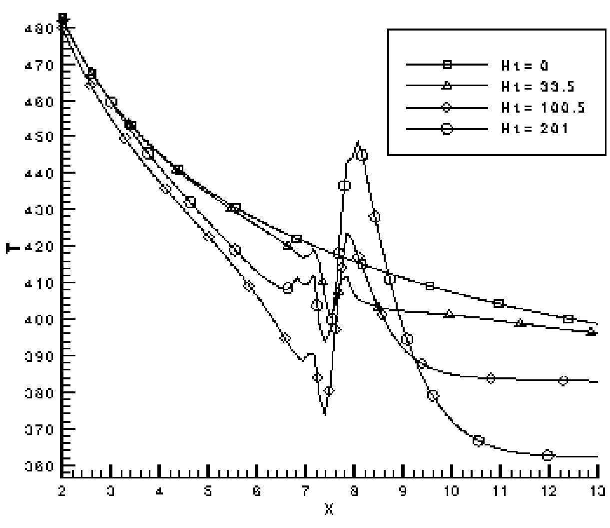


Figure 5: Temperature variation along fluid/solid interface

Effect of flow on magnetic field

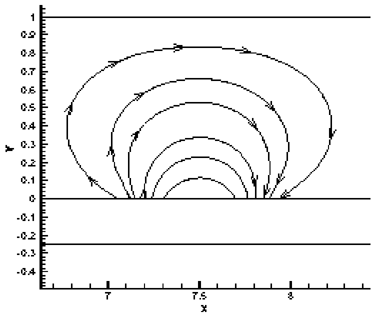


Figure 14: Magnetic field lines for $Rm = 2.7 \times 10^{-9}$

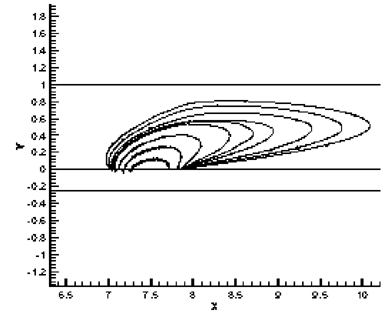


Figure 15: Magnetic field lines for $Rm = 20$

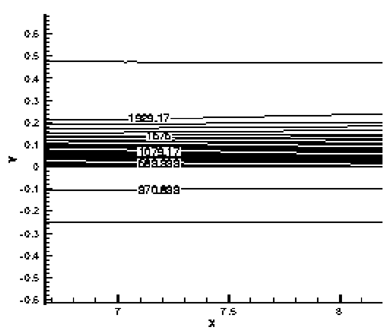


Figure 7: Temperature contours for $Ht = 0$

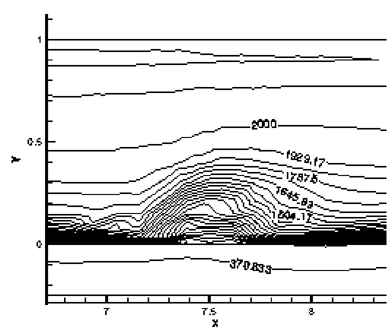


Figure 8: Temperature contours for $Ht = 201$

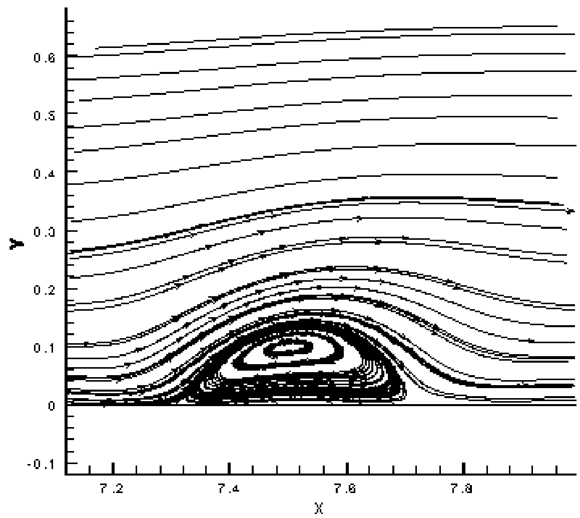


Figure 9: Vortex generated by applied magnetic field for $Ht = 102$

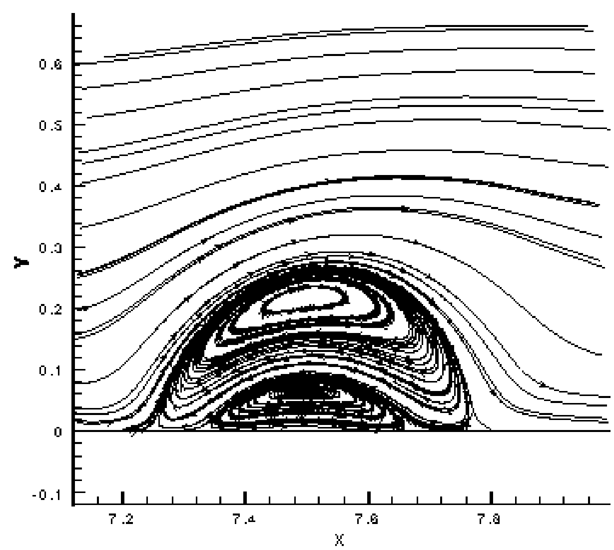


Figure 10: Vortex generated by applied magnetic field for $Ht = 201$

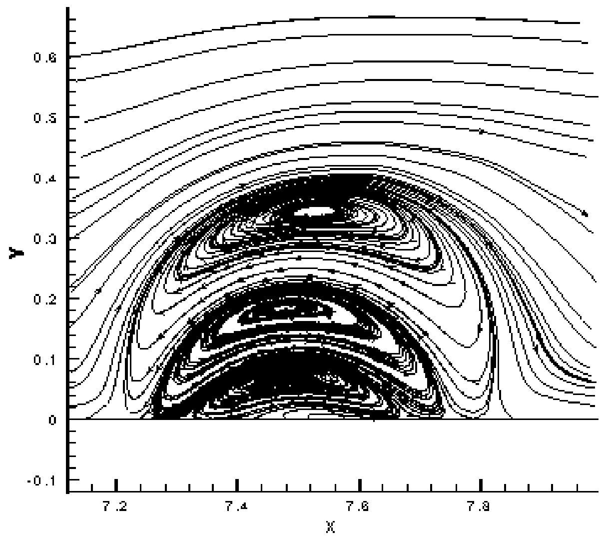


Figure 11: Vortex generated by applied magnetic field for $Ht = 335$

Simulation of Electro-magneto-hydrodynamic Flows

Objective

- Simulation of a steady state EMHD blood pump
- Both electric and magnetic fields are required to produce the driving force

Governing Equations

$$
\begin{aligned}
\nabla \cdot \mathbf{V} &= 0 && (20)\\
\mathbf{V} \cdot \nabla \mathbf{V} + \tfrac{1}{Re} \nabla \times \omega + \nabla P - \tfrac{Ht^2}{Re} \mathbf{V} \times \mathbf{B} \times \mathbf{B} & && (21)\\
- Se \hat{q} \mathbf{E} - M_1 \mathbf{V} \times \mathbf{B} \hat{q} - M_2 \mathbf{E} \times \mathbf{B} & &&\\
+ M_3 \nabla \hat{q} \times \mathbf{B} - M_4 \hat{q} \mathbf{E} \times \mathbf{B} &= 0 &&\\
\omega - \nabla \times \mathbf{V} = 0 & && (22)\\
\mathbf{V} \cdot \nabla T + \nabla \cdot \mathbf{q} - \tfrac{Ht^2 Ec}{Re} (\mathbf{V} \times \mathbf{B})^2 &= 0 && (23)\\
\mathbf{q} + \tfrac{1}{Pe} \nabla T &= 0 && (24)\\
\nabla \times \mathbf{q} &= 0 && (25)\\
\nabla \cdot \mathbf{B} &= 0 && (26)\\
\nabla \times \mathbf{B} = Rm \mathbf{V} \times \mathbf{B} + B_1 \mathbf{V} \hat{q} + B_2 \mathbf{E} - B_3 \nabla \hat{q} + B_4 \hat{q} \mathbf{E} & && (27)\\
\nabla \cdot \mathbf{E} &= Ne \hat{q} && (28)\\
\nabla \times \mathbf{E} &= 0 && (29)\\
\nabla \Phi &= \mathbf{E} && (30)\\
Q_1 \mathbf{V} \cdot \nabla \hat{q} + Q_2 \mathbf{E} \cdot \nabla \hat{q} + Q_3 \hat{q} + Q_4 \hat{q}^2 + Q_5 \nabla \cdot (\mathbf{V} \times \mathbf{B}) &= 0 && (31)
\end{aligned}
$$

Boundary conditions and geometry

•Rectangular domain with height of 4 cm and length of 40 cm

•Triangular mesh:

7021 nodes

3422 elements

parabolic triangles

•Specified parabolic inlet velocity profile and temperature of 310.15 K

•No slip on walls

•Wall temperature was 298.15 K

•Specified exit pressure of 1 Pa

•Positive electrode on bottom wall, negative electrode on top with 50 volts applied across them

•Uniform magnetic field of .05 Tesla specified in Z direction

Physical parameters for EMHD blood pump

Density(kg m^{-3}) = 1055.0

Inlet height(cm) = 4

Length(cm) = 40

Inlet temp.(K) = 310

Wall temp(K) = 298

heat conductivity(W kg^{-1} K^{-1}) = .51

specific heat(J kg^{-1} K^{-1})= 4178

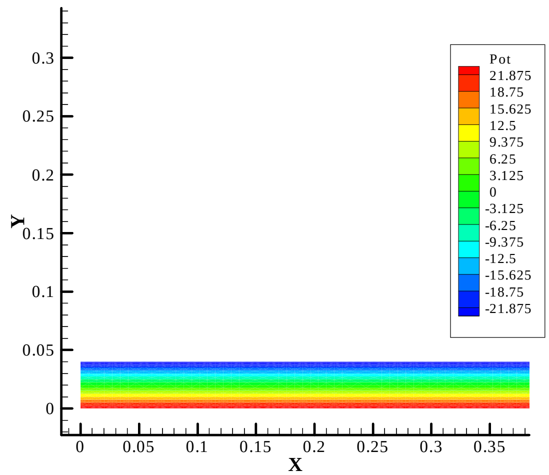
inlet velocity(m s^{-1}) = .05

dynamic viscosity(kg m^{-1} s^{-1}) = .004

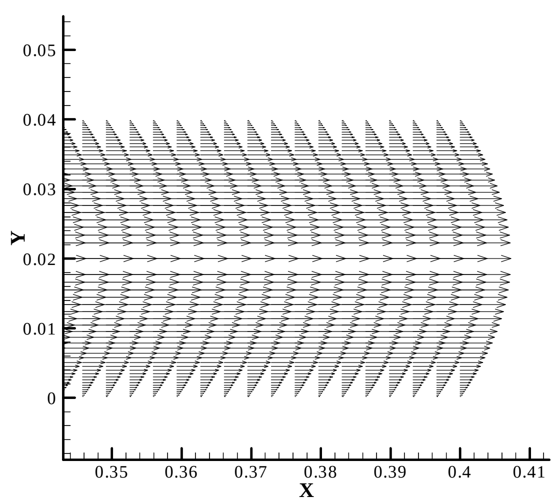
electric conductivity (S m^{-1}) = 1.4

outlet pressure (Pa) = 1

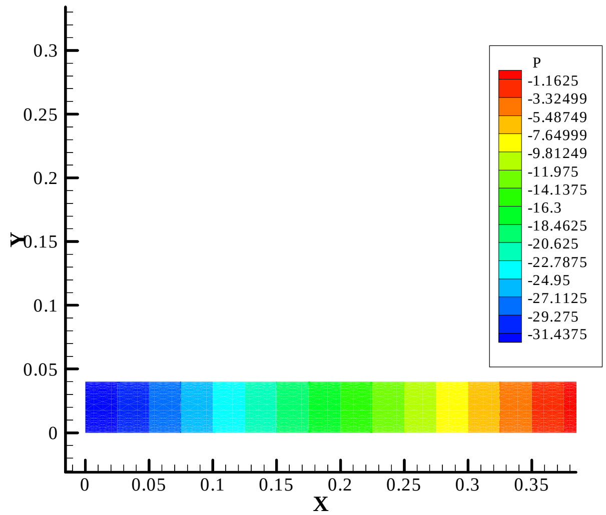
Computed Electric Potential



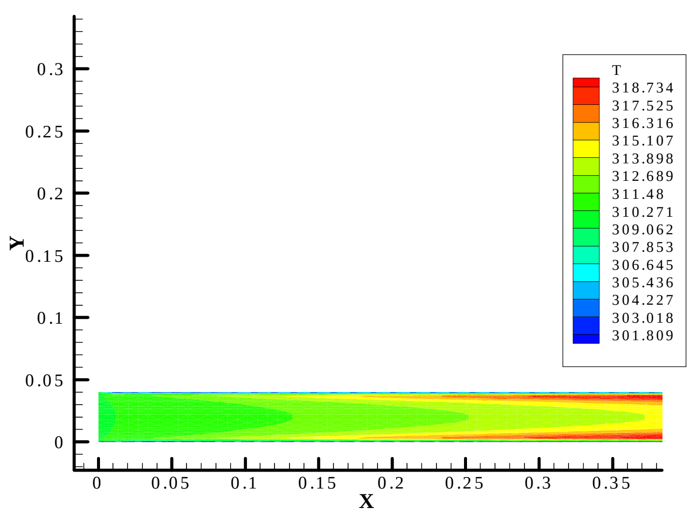
Computed Velocity Vectors



Computed Static Pressure

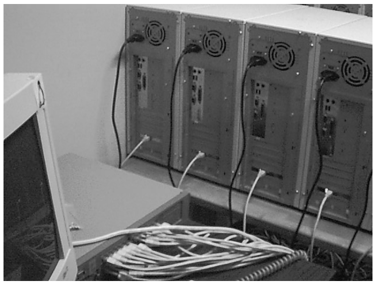


Computed Static Temperature



Parallel Computer

- based on commodity hardware components and public domain software
- 16 dual Pentium II 400 MHz based PC's
- 100 Megabits/second switched ethernet



Parallel Computer cont.

- total of 32 processors and 8.2 GB of main memory
- Compressible NSE solver achieved 1.5 Gflop/sec with a LU SSOR solver on a 100x100x100 structured grid
- GA optimization of a MHD diffuser completed in 30 hours. Same problem would take 14 days on a single CPU

Conclusions

- A code for the simulation of MHD/EMHD was developed based on the LSFEM
- Code was tested against analytic solutions and experimental data for separate disciplines
- Code was applied to several MHD problems including a MHD diffuser optimization problem
- Code was used to simulation an EMHD pump

Future Work

- Extension to axisymmetric configurations where crossed fields can be treated more realistically
- P-version
- Better preconditioning of linear systems
- Closer look at scaling of the equations